International Journal of Theoretical Physics, Vol. 7, No. 3 (1973), pp. 241–251

'The Concomitants of Spinors of Type [3/2, 1/2] in Space-Time'

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Received: 16 *May* 1972

Abstract

Explicit forms for the concomitants which are bilinear in two spinors of type [3/2, 1/2] and the concomitants which are quadratic in a single spinor of type [3/2, 1/2] are obtained. The dual-tensors, where they exist, are also given.

The concomitants of higher-order spinors can be obtained in an exactly similar manner.

1. *Introduction*

The definition of spinors is dependent upon the frame of anticommuting matrices. In four-dimensional space-time, where the metric is taken to have signature +2, the anticommuting set of matrices X_i is taken as that used by Littlewood (1972), namely

$$X_1=\begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{bmatrix},\quad X_2=\begin{bmatrix}&-1&&\\-1&&&\\&&&1\\&&1&\end{bmatrix},$$

$$X_3=\begin{bmatrix}&&&-1\\&&-1&\\&-1&&\\-1&&&\end{bmatrix},\quad X_0=\begin{bmatrix}&1&&\\-1&&&\\&&&1\\&&-1&\end{bmatrix},\qquad (1.1)$$

where

$$X_0{}^2=-I,\qquad X_i{}^2=I$$
$$\tilde{X}_0=-X_0,\qquad \tilde{X}_i=X_i\qquad (i=1,2,3).$$

Defining the metric tensor as

$$g_{ij}=g^{ij}=\begin{cases}0 & (i\neq j),\\ 1 & (i=j;\ i,j=1,2,3),\\ -1 & (i=j=0),\end{cases}$$

it follows that

$$X^0 = -X_0, \qquad X^i = X_i, \qquad (i = 1, 2, 3),$$

and hence

$$(x_i X^i)^2 = (x^i X_i)^2 = g_{ij} x^i x^j = g^{ij} x_i x_j,$$

which will be taken as the metric of space-time.

Corresponding to any Lorentz transformation, L say, as given by

$$x_i' = a_i{}^j x_j, \qquad (i = 1, 2, 3, 0),$$

there is a matrix U, the basic spin matrix, which is unique apart from sign such that

$$a_i{}^j X_j = U^{-1} X_i U,$$

where $[a_s{}^t]$ is a Lorentz matrix whose elements thus satisfy the 'orthogonal' relations

$$a^k{}_i a_{kj} = g_{ij}, \qquad a^{ki} a_k{}^j = g^{ij}. \tag{1.2}$$

A four-rowed real vector which, under L, is transformed by U, is called a basic spinor. By considering the direct product of a simple tensor of type $\{n\}$ and a basic spinor, on removal of the contractions present, an irreducible symmetric spinor of type $[n + 1/2, 1/2]$ is obtained, n being a positive integer. Explicit forms for such spinors have been given elsewhere (Dodds, 1972). If $V_{i_1 \ldots i_n}$ is an irreducible symmetric spinor of type $[n + 1/2, 1/2]$, it consists of $(n + 3)!/n!3!$ real four-vectors of which just $(n + 2)!/n!2!$ are independent because of the zero contractions $X^{i_1} V_{i_1 \ldots i_n} = 0 = g^{i_1 i_2} V_{i_1 \ldots i_n}$.

The concomitants which are bilinear in two spinors of type $[n + 1/2, 1/2]$ and the concomitants which are quadratic in a single spinor of type $[n + 1/2, 1/2]$, are of types given by the expansions of the products $[n + 1/2, 1/2][n + 1/2, 1/2]$ and $[n + 1/2, 1/2] \otimes \{2\}$ respectively. In the case of basic spinors, i.e. when $n = 0$, it is well known that the concomitants which are bilinear in two basic spinors are five in number, consisting of an invariant, a pseudo-invariant, a four-vector, a pseudo four-vector and a six-vector. When the basic spinors are made equal to give the concomitants which are quadratic in a single basic spinor, just the four-vector and the six-vector survive. This paper is concerned with the analysis of the higher-order case when $n = 1$, i.e. when the spinor or spinors are of type $[3/2, 1/2]$.

In order to illustrate later results, a particular reference frame is used. Suppose that W and Z are two basic spinors and that ξ_i and η_i are two tensors of type $\{1\}$. Putting $W_i = \xi_i W$ and $Z_i = \eta_i Z$, then V_i and Y_i, where $X^i V_i = 0 = X^i Y_i$, are two irreducible spinors of type $[3/2, 1/2]$ where (Dodds, 1972)

$$V_i = W_i - X_i(X^j W_j)/4, \qquad Y_i = Z_i - X_i(X^j Z_j)/4. \tag{1.3}$$

Following Littlewood (1969, 1972), the particular reference frame is chosen in which one of the basic spinors, Z say, is in canonical form. Hence, in this reference frame, the basic spinors are given by

$$\tilde{W} = [\alpha, \beta, \gamma, \delta], \qquad \tilde{Z} = [\varepsilon, 0, 0, 0],$$

where α, β, γ, δ and ε are real scalars. Writing

$$\tilde{W}_i = [\xi_i\alpha, \xi_i\beta, \xi_i\gamma, \xi_i\delta] = [\alpha_i, \beta_i, \gamma_i, \delta_i]$$

say, and similarly,

$$\tilde{Z}_i = [\varepsilon_i, 0, 0, 0],$$

using (1.1) and (1.3) it follows that

$$\tilde{V}_i = \tfrac{1}{4}[A_i, B_i, C_i, D_i]$$

say, where

$$\begin{aligned}
A_1 &= 3\alpha_1 + \beta_2 + \delta_3 + \beta_0, & A_2 &= -\beta_1 + 3\alpha_2 - \gamma_3 + \alpha_0,\\
B_1 &= 3\beta_1 - \alpha_2 - \gamma_3 + \alpha_0, & B_2 &= \alpha_1 + 3\beta_2 - \delta_3 - \beta_0,\\
C_1 &= 3\gamma_1 - \delta_2 + \beta_3 + \delta_0, & C_2 &= \delta_1 + 3\gamma_2 + \alpha_3 - \gamma_0,\\
D_1 &= 3\delta_1 + \gamma_2 - \alpha_3 + \gamma_0, & D_2 &= -\gamma_1 + 3\delta_2 + \beta_3 + \delta_0,\\
A_3 &= -\delta_1 + \gamma_2 + 3\alpha_3 + \gamma_0, & A_0 &= \beta_1 + \alpha_2 + \gamma_3 + 3\alpha_0,\\
B_3 &= \gamma_1 + \delta_2 + 3\beta_3 - \delta_0, & B_0 &= \alpha_1 - \beta_2 - \delta_3 + 3\beta_0,\\
C_3 &= -\beta_1 - \alpha_2 + 3\gamma_3 + \alpha_0, & C_0 &= \delta_1 - \gamma_2 + \alpha_3 + 3\gamma_0,\\
D_3 &= \alpha_1 - \beta_2 + 3\delta_3 - \beta_0, & D_0 &= \gamma_1 + \delta_2 - \beta_3 + 3\delta_0,
\end{aligned} \tag{1.4}$$

and

$$Y_1 = \tfrac{1}{4}\begin{bmatrix} 3\varepsilon_1 \\ -\varepsilon_2 + \varepsilon_0 \\ 0 \\ -\varepsilon_3 \end{bmatrix}, \qquad Y_2 = \tfrac{1}{4}\begin{bmatrix} 3\varepsilon_2 + \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_3 \\ 0 \end{bmatrix}, \qquad Y_3 = \tfrac{1}{4}\begin{bmatrix} 3\varepsilon_3 \\ 0 \\ -\varepsilon_2 + \varepsilon_0 \\ \varepsilon_1 \end{bmatrix},$$

$$Y_0 = \tfrac{1}{4}\begin{bmatrix} \varepsilon_2 + 3\varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_3 \\ 0 \end{bmatrix}.$$

Also, since $X^i V_i = 0$, it follows that

$$\begin{aligned}
A_1 - B_2 - D_3 - B_0 &= 0, & -B_1 - A_2 - C_3 + A_0 &= 0,\\
C_1 + D_2 - B_3 - D_0 &= 0, & -D_1 + C_2 - A_3 + C_0 &= 0.
\end{aligned} \tag{1.5}$$

2.

Suppose that $L(+,+)$, $L(+,-)$, $L(-,+)$ and $L(-,-)$ represent Lorentz transformations belonging to the four separate 'pieces' of the Lorentz group, the first sign denoting the sign of the determinant of the transforming matrix, whilst the second sign denotes whether the transformation leaves unchanged (+) or reverses (−) the direction of the time-axis. Because of the differing properties portrayed by the basic spin matrix corresponding to a transformation $L(+,+)$, $L(+,-)$, $L(-,+)$ and $L(-,-)$, (see, for example, Dodds, 1971b), it is convenient, initially, to consider a transformation

$L(+,+)$, the extension to the full Lorentz group being given at the conclusion of the analysis.

Consider then a transformation $L(+,+)$, $x_i' = a_i{}^j x_j$ say. The corresponding basic spin matrix is U where

$$a_i{}^j X_j = U^{-1} X_i U \tag{2.1}$$

and (Dodds, 1971b)

$$\begin{aligned} U^{-1} &= -T\tilde{U}T \\ \tilde{U}TU &= T, \qquad U\phi = \phi U, \end{aligned} \tag{2.2}$$

where $X_0 = T$ and $X_1 X_2 X_3 X_0 = \phi$. Under $L(+,+)$,

$$V_i \to U a_i{}^j V_j, \qquad Y_i \to U a_i{}^j Y_j.$$

The types of the second-order concomitants are determined by 'characteristic analysis', for a full account of the methods of which one is referred to Littlewood, 1944, and 1950, p. 288 et seq. Since

$$\{1\}\Delta = [3/2, 1/2] + \Delta,$$

where $\Delta = [1/2, 1/2]$, it follows that

$$[3/2, 1/2] = (\{1\} - 1)\Delta.$$

The types of the concomitants which are bilinear in two spinors of type [3/2, 1/2] correspond to the product [3/2, 1/2][3/2, 1/2], where

$$\begin{aligned} [3/2, 1/2][3/2, 1/2] &= (\{1\} - 1)(\{1\} - 1)\Delta^2 \\ &= (\{2\} + \{1^2\} - 2\{1\} + 1)(2\{0\} + 2\{1\} + \{1^2\}) \\ &= \{31\} + \{2^2\} + 2\{3\} + 2\{21\} + \{1^2\} + \{0\} - 2\{1\} \\ &\quad (\text{since } \{21^2\} = \{2\} + \{1^2\} - \{0\} \text{ and } \{1^4\} = \{0\} \text{ here}) \\ &= [31] + [2^2] + 2[3] + 2[21] + 2[2] + 2[1^2] + 2[1] + 2[0]. \end{aligned} \tag{2.3}$$

As a check, the L.H.S. contains $(12)^2 = 144$ terms, whilst the R.H.S. contains $30 + 10 + 2(16) + 2(16) + 2(9) + 2(6) + 2(4) + 2(1) = 144$ terms as required.

The types of the concomitants which are quadratic in a single spinor of type [3/2, 1/2] correspond to the product $[3/2, 1/2] \otimes \{2\}$, where

$$\begin{aligned} [3/2, 1/2] \otimes \{2\} &= (\{1\}\Delta - \Delta) \otimes \{2\} \\ &= (\{1\}\Delta) \otimes \{0\}\Delta \otimes \{1^2\} - (\{1\}\Delta) \otimes \{1\}\Delta \otimes \{1\} \\ &\qquad + (\{1\}\Delta) \otimes \{2\}\Delta \otimes \{0\} \\ &= \Delta \otimes \{1^2\} - \{1\}\Delta^2 + (\{1\}\Delta) \otimes \{2\} \\ &= \Delta \otimes \{1^2\} - \{1\}\Delta^2 + \{1\} \otimes \{2\}\Delta \otimes \{2\} \\ &\qquad + \{1\} \otimes \{1^2\}\Delta \otimes \{1^2\} \\ &= 2\{0\} + \{1\} - \{1\}(2\{0\} + 2\{1\} + \{1^2\}) + \{2\}(\{1\} + \{1^2\}) \\ &\qquad + \{1^2\}(2\{0\} + \{1\}) \\ &= \{31\} + \{3\} + \{21\} + \{1^2\} + \{0\} - \{2\} - \{1\} \\ &= [31] + [3] + [21] + 2[1^2] + [1]. \end{aligned} \tag{2.4}$$

As a check, the L.H.S. contains $(12!/1!11!)^2-(12!/2!10!)=78$ terms, whilst the R.H.S. contains $30+16+16+2(6)+4=78$ terms as required.

Having obtained the types of the second-order concomitants, it remains to determine explicit forms for them. This is done by considering quantities of various rank which are bilinear in the spinors Y_i and V_i. In these considerations, one is only interested in those quantities which are both non-zero and irreducible, quantities which are zero or reducible existing because of the zero contractions $X^i V_i = 0 = X^i Y_i$, from which follow the formulae

$$\tilde{V}^k TX_i X_k = 2\tilde{V}_i T, \quad \tilde{V}^k TX_i X_j X_k = 2(\tilde{V}_j TX_i - \tilde{V}_i TX_j)$$
$$X^k X_i V_k = 2V_i, \quad X^k X_i X_j V_k = 2(X_j V_i - X_i V_j), \quad (i, j = 1,2,3,0), \tag{2.5}$$

together with similar formulae for Y_i. Note that the matrix ϕ may be introduced into the above formulae, since ϕ anticommutes with each of the X_i.

Quantities of rank zero are considered first. There are just two such quantities, non-zero and irreducible, viz. $\tilde{Y}^k TV_k$ and $\tilde{Y}^k T\phi V_k$. After transformation,

$$\tilde{Y}^k TV_k \to a^{ki} a_k{}^j \tilde{Y}_i \tilde{U} TUV_j = g^{ij} \tilde{Y}_i TV_j = \tilde{Y}^k TV_k,$$

where (1.2) and (2.2) have been used. Similarly,

$$\tilde{Y}^k T\phi V_k \to \tilde{Y}^k T\phi V_k$$

since, from (2.2), $\phi U = U\phi$. Hence both $\tilde{Y}^k TV_k$ and $\tilde{Y}^k T\phi V_k$ are complete tensors of rank zero.

Considering next quantities of rank one, there are just two such quantities, viz. $\tilde{Y}^p TX_i V_p$ and $\tilde{Y}^p TX_i \phi V_p$. After transformation,

$$\tilde{Y}^p TX_i V_p \to a^{pj} a_p{}^k \tilde{Y}_j \tilde{U} TX_i UV_k = g^{jk} \tilde{Y}_j \tilde{U} TUa_i{}^r X_r V_k = a_i{}^r \tilde{Y}^p TX_r V_p,$$

where (1.2), (2.1) and (2.2) have been used. Similarly, $\tilde{Y}^p TX_i \phi V_p \to a_i{}^r \tilde{Y}^p TX_r \phi V_p$, and hence both $\tilde{Y}^p TX_i V_p$ and $\tilde{Y}^p TX_i \phi V_p$ are complete tensors of rank one.

Considering next quantities of rank two, there are just four such quantities, viz. $\tilde{Y}^p TX_i X_j V_p$, $\tilde{Y}^p TX_i X_j \phi V_p$, $\tilde{Y}_i TV_j$ and $\tilde{Y}_i T\phi V_j$, each of which, in the manner of the preceding cases, is found to be a complete tensor of rank two. Similarly there are just two complete tensors of rank three, viz. $\tilde{Y}_i TX_j V_k$ and $\tilde{Y}_i TX_j \phi V_k$, and just two complete tensors of rank four, viz. $\tilde{Y}_i TX_j X_k V_p$ and $\tilde{Y}_i TX_j X_k \phi V_p$. All quantities of rank >4 are either identically equal to zero or are reducible to quantities already considered of rank $\leqslant 4$.

Twelve complete tensors of various ranks are thus obtained from which the concomitants are to be constructed. In constructing simple tensors from these complete tensors, since one is in four dimensions, it is only necessary to consider partitions into not more than two parts. It is convenient to write, for example, 'the concomitant of type [31](ijk,p)' as shorthand for 'the concomitant of type [31] corresponding to the standard Young Tableau $\begin{pmatrix} i\,j\,k \\ p \end{pmatrix}$'.

The concomitants which are linear in each of Y_i and V_i are considered first. They are fourteen in number, their types being given by the character equation (2.3). The concomitants of type [0] and [1] are just the complete tensors of ranks zero and one respectively. The concomitants of types [2] and $[1^2]$ are obtained from the complete tensors of rank two by operating on the tensors with the necessary symmetrising operators to give simple tensors of types {2} and $\{1^2\} = [1^2]$. Removal of the contraction with the metric tensor from the simple tensors of type {2} will yield simple tensors of type [2] as required. Note that since $X_i X_j$ is skew-symmetric, no term of type [2] is obtained from either $\tilde{Y}^p TX_i X_j V_p$ or $\tilde{Y}^p TX_i X_j \phi V_p$. Further, since $X_i X_j \phi$ is exactly and precisely the dual of $X_i X_j$, when the tensors of type $[1^2]$ are being constructed, it is not necessary to consider both of the complete tensors $\tilde{Y}^p TX_i X_j V_p$ and $\tilde{Y}^p TX_i X_j \phi V_p$ since the tensor of type $[1^2]$ constructed from the latter will be simply the dual of the tensor of type $[1^2]$ constructed from the former. The concomitants of types [3] and [21] are constructed from the complete tensors of rank three by a similar process of symmetrisation and contraction with the metric tensor. Note that corresponding to the partition (21), there are two standard Young Tableaux, viz. (ij,k) and (ik,j), and hence two corresponding symmetrising operators (see, for example, Dodds, 1971a, Appendix). Finally the concomitants of type [31] and $[2^2]$ are constructed from the complete tensors of rank four, viz. $\tilde{Y}_i TX_j X_k V_p$ and $\tilde{Y}_i TX_j X_k \phi V_p$. Note firstly that there is no concomitant of type [4] because $X_j X_k$ is skew-symmetric. Secondly, in constructing the concomitants of type [31] and $[2^2]$, it is only necessary to consider one of the complete tensors of rank four, $\tilde{Y}_i TX_j X_k V_p$ say, since the second one will merely yield the duals of the tensors of types [31] and $[2^2]$ constructed from $\tilde{Y}_i TX_j X_k V_p$, $X_j X_k \phi$ being the dual of $X_j X_k$ as mentioned previously. Thus the concomitants of types [31] and $[2^2]$ are constructed from $\tilde{Y}_i TX_j X_k V_p$ by the process of symmetrisation and contraction with the metric tensor. Note that corresponding to the partitions (31) and (2^2) there are three and two standard Young Tableaux respectively, viz. (ijk,p), (ijp,k), (ikp,j), and (ij,kp), (ik,jp), and hence three and two corresponding symmetrising operators (see Dodds, 1971a, Appendix). It is found however that the concomitant of type [31](ijk,p) is identically zero. A final note, formulae (2.5) are used frequently in removing the contractions with the metric tensor from the various tensors. The concomitants which are linear in each of Y_i and V_i are thus as follows:

$$[0]: F = \tilde{Y}^q TV_q, \quad {}^+F; \qquad [1]: G_i = \tilde{Y}^q TX_i V_q, \quad {}^+G_i;$$

$$[2]: J_{ij} = P_{ij}(\tilde{Y}_i TV_j - \tfrac{1}{4} g_{ij} F), \quad {}^+J_{ij};$$

$$[1^2]: H_{ij} = \tfrac{1}{2}(\tilde{Y}_i T V_j - \tilde{Y}_j T V_i), \quad {}^+H_{ij}, \quad I_{ij} = \tfrac{1}{2}\tilde{Y}^q T(X_i X_j - X_j X_i) V_q;$$

$$[3]: M_{ijk} = P_{ijk}(\tilde{Y}_i T X_j V_k - \tfrac{1}{6} g_{ij} G_k), \quad {}^+M_{ijk};$$

$$[21]: K_{ijk} = \tfrac{1}{3}(\tilde{Y}_i T X_j V_k - \tilde{Y}_k T X_x V_i + \tilde{Y}_j T X_i V_k - \tilde{Y}_k T X_i V_j), \quad {}^+K_{ijk},$$

$$L_{ijk} = \tfrac{1}{3}(\tilde{Y}_i T X_j V_k - \tilde{Y}_j T X_i V_k + \tilde{Y}_k T X_j V_i - \tilde{Y}_j T X_k V_i), \quad {}^+L_{ijk}; \qquad (2.6)$$

$$\begin{aligned}
[31]: R_{ijkp} = {} & \tfrac{1}{8}(\tilde{Y}_i T X_j X_k V_p + \tilde{Y}_p T X_j X_k V_i + \tilde{Y}_p T X_i X_k V_j \\
& + \tilde{Y}_j T X_i X_k V_p + \tilde{Y}_j T X_p X_k V_i + \tilde{Y}_i T X_p X_k V_j) - \tfrac{1}{4}(g_{ki} J_{jp} \\
& + g_{kj} J_{pi} + g_{kp} J_{ij}) + \tfrac{1}{12}(g_{ij} H_{kp} + g_{jp} H_{ki} + g_{pi} H_{kj}) \\
& + \tfrac{1}{24}(g_{ij} I_{kp} + g_{jp} I_{ki} + g_{pi} I_{kj}) \\
& - \tfrac{1}{16}(g_{ij} g_{kp} + g_{jp} g_{ki} + g_{pi} g_{kj}) F, \\
S_{ijkp} = {} & \tfrac{1}{8}(\tilde{Y}_i T X_j X_p V_k + \tilde{Y}_k T X_j X_p V_i + \tilde{Y}_k T X_j X_i V_p \\
& + \tilde{Y}_p T X_j X_i V_k + \tilde{Y}_p T X_j X_k V_i + \tilde{Y}_i T X_j X_k V_p) \\
& - \tfrac{1}{4}(g_{ji} J_{kp} + g_{jk} J_{pi} + g_{jp} J_{ik}) - \tfrac{1}{12}(g_{ik} H_{jp} + g_{kp} H_{ji} \\
& + g_{pi} H_{jk}) - \tfrac{1}{24}(g_{ik} I_{jp} + g_{kp} I_{ji} + g_{pi} I_{jk}) \\
& - \tfrac{1}{16}(g_{ki} g_{jp} + g_{ip} g_{jk} + g_{pk} g_{ji}) F; \\
[2^2]: N_{ijkp} = {} & \tfrac{1}{12}(\tilde{Y}_i T X_j X_k V_p + \tilde{Y}_i T X_j X_p V_k + \tilde{Y}_j T X_i X_k V_p \\
& + \tilde{Y}_j T X_i X_p V_k + \tilde{Y}_k T X_p X_i V_j + \tilde{Y}_p T X_k X_i V_j \\
& + \tilde{Y}_k T X_p X_j V_i + \tilde{Y}_p T X_k X_j V_i) \\
& + \tfrac{1}{3}(g_{ij} J_{kp} + g_{kp} J_{ij}) - \tfrac{1}{3}(g_{jk} J_{ip} + g_{ki} J_{pj} + g_{ip} J_{jk} + g_{pj} J_{ki}) \\
& + \tfrac{1}{18}(g_{ij} g_{kp} - 2 g_{jk} g_{ip} - 2 g_{ki} g_{jp}) F, \\
Q_{ijkp} = {} & \tfrac{1}{12}(\tilde{Y}_i T X_j X_k V_p + \tilde{Y}_j T X_i X_p V_k + \tilde{Y}_i T X_p X_k V_j \\
& + \tilde{Y}_p T X_i X_j V_k + \tilde{Y}_k T X_j X_i V_p + \tilde{Y}_j T X_k X_p V_i \\
& + \tilde{Y}_k T X_p X_i V_j + \tilde{Y}_p T X_k X_j V_i) \\
& - \tfrac{1}{3}(g_{jp} J_{ki} + g_{ki} J_{jp}) - \tfrac{1}{18}(g_{ij} g_{kp} + g_{jk} g_{ip} + g_{ki} g_{jp}) F;
\end{aligned}$$

where K_{ijk} and ${}^+K_{ijk}$ are of type $[21](ij,k)$, L_{ijk} and ${}^+L_{ijk}$ are of type $[21](ik,j)$, whilst R_{ijkp}, S_{ijkp}, N_{ijkp} and Q_{ijkp} are of types $[31](ijp,k)$, $[31](ikp,j)$, $[2^2](ij,kp)$ and $[2^2](ik,jp)$ respectively.

Firstly, note that permutation operators P appear in the forms given for the concomitants of types [2] and [3], the operators being defined as follows: A permutation operator $P_{i_1 \ldots i_n}$ operating on a tensor $\Gamma_{i_1 \ldots i_n}$ say denotes the operation of taking the arithmetic mean of the $n!$ tensors obtained by permuting the suffixes of $\Gamma_{i_1 \ldots i_n}$ in all ways. Note also that the formula $X_i X_j + X_j X_i = 2g_{ij}$ has been used in expressing R_{ijkp}, S_{ijkp}, N_{ijkp} and Q_{ijkp} in the forms given. This explains the presence of the F terms in R_{ijkp} and S_{ijkp}. Finally, the operation denoted by '+' on a tensor requires the insertion of ϕ immediately prior to a V_i term wherever such a term is present in the tensor, e.g. ${}^+G_i = \tilde{Y}^q T X_i \phi V_q$. Since $\phi^2 = -1$, the operation '++' on a tensor gives the original tensor together with a sign reversal, e.g. $^{++}F = -F$. As noted previously, the tensors $^+I_{ij}$, $^+R_{ijkp}$, $^+S_{ijkp}$, $^+N_{ijkp}$ and $^+Q_{ijkp}$ are just the dual tensors of I_{ij}, R_{ijkp}, S_{ijkp}, N_{ijkp} and Q_{ijkp} respectively, the dual tensors, where they exist, being defined shortly.

The characteristic analysis required fourteen concomitants, whereas nineteen, as given by (2.6), have been found. It is necessary to define certain dual tensors in order to resolve this discrepancy: Define the dual tensors of H_{ij}, I_{ij}, K_{ijk}, L_{ijk}, R_{ijkp}, S_{ijkp}, N_{ijkp} and Q_{ijkp} as $^*H_{ij}$, $^*I_{ij}$, $^*K_{ijk}$, $^*L_{ijk}$, $^*R_{ijkp}$, $^*S_{ijkp}$, $^*N_{ijkp}$ and $^*Q_{ijkp}$ respectively, where

$$^*H_{ij} = \tfrac{1}{2}E_{ijqr}H^{qr}, \qquad ^*K_{ijk} = \tfrac{1}{3}(g_{iq}E_{jkrs} + g_{jq}E_{ikrs})K^{qrs},$$

$$^*I_{ij} = \tfrac{1}{2}E_{ijqr}I^{qr}, \qquad ^*L_{ijk} = \tfrac{1}{3}(g_{iq}E_{jkrs} + g_{kq}E_{jirs})L^{qrs},$$

$$^*R_{ijkp} = \tfrac{1}{4}(g_{iq}g_{js}E_{kprs} + g_{jq}g_{pt}E_{kirs} + g_{pq}g_{it}E_{kjrs})R^{qrst},$$

$$^*S_{ijkp} = \tfrac{1}{4}(g_{iq}g_{kt}E_{jprs} + g_{kq}g_{pt}E_{jirs} + g_{pq}g_{it}E_{jkrs})S^{qrst},$$

$$^*N_{ijkp} = \tfrac{1}{6}(g_{iq}g_{kt}E_{jprs} + g_{jq}g_{kt}E_{iprs} + g_{iq}g_{pt}E_{jkrs} + g_{jq}g_{pt}E_{ikrs})N^{qrst},$$

$$^*Q_{ijkp} = \tfrac{1}{6}(g_{iq}g_{jt}E_{kprs} + g_{kq}g_{jt}E_{iprs} + g_{iq}g_{pt}E_{kjrs} + g_{kq}g_{pt}E_{ijrs})Q^{qrst},$$

and E_{ijkp} is the alternating tensor, which is defined to be equal to $+1$ if (i,j,k,p) is a positive permutation of $(1,2,3,0)$, equal to -1 if a negative permutation and equal to zero if any suffix is repeated. The dual tensor of a tensor is of course of the same type as the original tensor. The above dual tensors are easily seen to define tensors of the required types. The dual tensors $^{*+}H_{ij}$, $^{*+}I_{ij}$, etc. are defined in a similar manner, e.g. $^{*+}H_{ij} = \tfrac{1}{2}E_{ijqr}{}^+H^{qr}$. Clearly, the operations denoted by '+' and '*' on a tensor are commutative and thus, for example, $^{*+}H_{ij} = {}^{+*}H_{ij}$. Further, the operation '**', where applicable, on a tensor gives the original tensor together with a sign reversal:

Consider K_{ijk}, then using the definition for $^*K_{ijk}$,

$$\begin{aligned} ^{**}K_{ijk} &= \tfrac{1}{3}(g_{iq}E_{jkrs} + g_{jq}E_{ikrs})\,^*K^{qrs} \\ &= \tfrac{1}{3}(g_{iq}E_{jkrs} + g_{jq}E_{ikrs})g^{qq'}g^{rr'}g^{ss'}\,^*K_{q'r's'} \\ &= \tfrac{1}{9}(g_{iq}E_{jkrs} + g_{jq}E_{ikrs})g^{qq'}g^{rr'}g^{ss'}(g_{q'l}E_{r's'mn} + g_{r'l}E_{q's'mn})K^{lmn} \\ &= \tfrac{1}{9}\{-2g_{il}(g_{jm}g_{kn} - g_{jn}g_{km}) + g^{ss'}E_{jkls}E_{is'mn} \\ &\qquad - 2g_{jl}(g_{im}g_{kn} - g_{in}g_{km}) + g^{ss'}E_{ikls}E_{js'mn}\}K^{lmn} \end{aligned}$$

since

$$\begin{cases} g_{iq}E_{jkrs}g^{qq'}g^{rr'}g^{ss'}g_{q'l}E_{r's'mn} = g_{il}g^{rr'}g^{ss'}E_{rsjk}E_{r's'mn} \\ \qquad\qquad = -2g_{il}(g_{jm}g_{kn} - g_{jn}g_{km}), \\ g_{iq}E_{jkrs}g^{qq'}g^{rr'}g^{ss'}g_{r'l}E_{q's'mn} = g^{ss'}E_{jkls}E_{is'mn}, \\ \text{and similarly for the other two terms.} \end{cases}$$

Further,

$$\begin{aligned} g^{ss'}E_{jkls}E_{is'mn}K^{lmn} &= g^{ss'}E_{sjkl}E_{s'imn}K^{lmn} \\ &= -(g_{jl}g_{km}g_{in} + g_{jm}g_{kn}g_{il} + g_{jn}g_{kl}g_{im} - g_{ji}g_{kn}g_{lm} - g_{jm}g_{kl}g_{in} \\ &\qquad - g_{jn}g_{km}g_{il})K^{lmn} \\ &= -(g_{jm}g_{kn}g_{il} - g_{jn}g_{km}g_{il})K^{lmn}, \end{aligned}$$

and similarly,

$$g^{ss'} E_{lkls} E_{js'mn} K^{lmn} = -(g_{lm} g_{kn} g_{ij} - g_{in} g_{km} g_{ij}) K^{lmn}.$$

Hence,

$$\begin{aligned} **K_{ijk} &= \tfrac{1}{9}\{-2g_{il}(g_{jm} g_{kn} - g_{jn} g_{km}) - (g_{jm} g_{kn} g_{il} - g_{jn} g_{km} g_{il}) \\ &\quad - 2g_{jl}(g_{im} g_{kn} - g_{in} g_{km}) - (g_{im} g_{kn} g_{lj} - g_{in} g_{km} g_{lj})\} K^{lmn} \\ &= \tfrac{1}{9}(-2K_{ijk} + 2K_{ikj} - K_{ijk} + K_{ikj} - 2K_{jik} + 2K_{jki} - K_{jik} + K_{jki}). \end{aligned}$$

But, since K_{ijk} is of type $[21](ij,k)$, $K_{ijk} = K_{jik}$ and $K_{ijk} + K_{jki} + K_{kij} = 0$. Hence,

$$**K_{ijk} = -K_{ijk} \tag{2.7}$$

as required. Corresponding results for other concomitants, where applicable, can be proved in a similar manner.

Using the particular reference frame described in the introduction, together with the various definitions, the following dependency relations are obtained:

$$\begin{array}{ll} I_{ij} = 2(H_{ij} - {}^{*+}H_{ij}), & {}^*I_{ij} = 2({}^*H_{ij} + {}^+H_{ij}), \\ K_{ijk} + {}^{*+}K_{ijk} = 2L_{ikj}, & {}^*K_{ijk} - {}^+K_{ijk} = 2{}^*L_{ikj}, \\ L_{ijk} = {}^{*+}L_{ijk}, & {}^*L_{ijk} = -{}^+L_{ijk}, \\ R_{ijkp} + S_{ikjp} = 0, & {}^*R_{ijkp} + {}^*S_{ikjp} = 0, \\ N_{ijkp} + Q_{ikjp} = 0, & {}^*N_{ijkp} + {}^*Q_{ikjp} = 0, \end{array} \tag{2.8}$$

where the relations on the right are the duals of the corresponding relations on the left. In illustration, consider the second relation on the right, viz. ${}^*K_{ijk} - {}^+K_{ijk} - 2{}^*L_{ikj} = 0$. Take $i = 1$, $j = 0$ and $k = 3$ say. By definition,

$${}^*K_{103} = \tfrac{1}{3}(E_{03rs} K^{1rs} - E_{13rs} K^{0rs}) = \tfrac{1}{3}(-K_{112} + K_{121} + K_{020} - K_{002}).$$

But

$$K_{121} = -\tfrac{1}{2}K_{112} \qquad \text{and} \qquad K_{020} = -\tfrac{1}{2}K_{002}.$$

Hence,

$${}^*K_{103} = -\tfrac{1}{2}(K_{112} + K_{002}).$$

Similarly,

$${}^*L_{130} = L_{112} + L_{002}.$$

Hence,

$$\begin{aligned} {}^*K_{103} - {}^+K_{103} - 2{}^*L_{130} &= -\tfrac{1}{2}(K_{112} + K_{002}) - {}^+K_{103} - 2(L_{112} + L_{002}) \\ &= -\tfrac{1}{4}\{(\tilde{Y}_1 TX_1 V_2 - \tilde{Y}_2 TX_1 V_1) + (-\tilde{Y}_0 V_2 + \tilde{Y}_2 V_0) \\ &\quad + (\tilde{Y}_1 TX_0 \phi V_3 - \tilde{Y}_3 TX_0 \phi V_1 + \tilde{Y}_0 TX_1 \phi V_3 - \tilde{Y}_3 TX_1 \phi V_0) \\ &\quad + 2(\tilde{Y}_2 TX_1 V_1 - \tilde{Y}_1 TX_2 V_1) + 2(\tilde{Y}_2 TX_0 V_0 - \tilde{Y}_0 TX_2 V_0)\} \\ &= \tfrac{1}{4}\{(-A_1 + B_2 + D_3 + B_0)\varepsilon_1 + (B_1 + A_2 + C_3 - A_0)\varepsilon_0\} \\ &= 0 \qquad \text{(using (1.5))}. \end{aligned}$$

Other suffixes may be chosen giving a similar zero result, and hence ${}^*K_{ijk} - {}^+K_{ijk} - 2{}^*L_{ikj} = 0$ as required.

In the light of dependency relations (2.8), it is seen that just fourteen of the nineteen concomitants, as given by (2.6), are independent, thus agreeing with the earlier characteristic analysis. The fourteen concomitants are taken to be

$$F,\quad {}^{+}F,\quad G_i,\quad {}^{+}G_i,\quad H_{ij},\quad {}^{+}H_{ij},\quad J_{ij},\quad {}^{+}J_{ij}$$
$$K_{ijk},\quad {}^{+}K_{ijk},\quad M_{ijk},\quad {}^{+}M_{ijk},\quad N_{ijkp},\quad R_{ijkp};$$

as given explicitly in (2.6).

The concomitants which are quadratic in a single spinor of type [3/2, 1/2] are now determined. From the earlier characteristic analysis, (2.4), they are six in number, one each of types [31], [3], [21], [1] and two of type $[1^2]$. Explicit forms for these concomitants are obtained, quite simply, by putting $Y_i = V_i$ in each of the concomitants which are bilinear in Y_i and V_i just obtained. Since

$$[3/2, 1/2] \otimes \{2\} = [31] + [3] + [21] + 2[1^2] + [1]$$

and

$$[3/2, 1/2] \otimes \{1^2\} = [2^2] + [3] + [21] + 2[2] + [1] + 2[0]$$

the character equation (2.3) may be rewritten in the form

$$[3/2, 1/2][3/2, 1/2] = [3/2, 1/2] \otimes \{2\} + [3/2, 1/2] \otimes \{1^2\}.$$

The concomitants which correspond to $[3/2, 1/2] \otimes \{2\}$ are symmetric in the two groundforms and hence survive when the groundforms are made equal. On the other hand, the concomitants which correspond to $[3/2, 1/2] \otimes \{1^2\}$ are skew-symmetric in the groundforms and will thus become zero when the groundforms are made equal. It is easily seen that the concomitants which are skew-symmetric in the spinors are F, ${}^{+}F$, ${}^{+}G_i$, J_{ij}, ${}^{+}J_{ij}$, K_{ijk}, ${}^{+}M_{ijk}$ and N_{ijkp}, all of which will thus become identically zero when the spinors are made equal. The concomitants which are quadratic in a single spinor of type [3/2, 1/2] are thus as follows:

$$
\begin{aligned}
&[1]: G_i = \tilde{V}^q T X_i V_q; \qquad [1^2]: H_{ij} = \tfrac{1}{2}(\tilde{V}_i T V_j - \tilde{V}_j T V_i), \quad {}^{+}H_{ij};\\
&[21]: {}^{+}K_{ijk} = \tfrac{1}{3}(\tilde{V}_i T X_j \phi V_k - \tilde{V}_k T X_j \phi V_i + \tilde{V}_j T X_i \phi V_k - \tilde{V}_k T X_i \phi V_j);\\
&[3]: M_{ijk} = P_{ijk}(\tilde{V}_i T X_j V_k - \tfrac{1}{6} g_{ij} G_k); \qquad\qquad (2.9)\\
&[31]: R_{ijkp} = \tfrac{1}{8}(\tilde{V}_i T X_j X_k V_p + \tilde{V}_p T X_j X_k V_i + \tilde{V}_p T X_i X_k V_j + \tilde{V}_j T X_i X_k V_p\\
&\qquad + \tilde{V}_j T X_p X_k V_i + \tilde{V}_i T X_p X_k V_j)\\
&\qquad + \tfrac{1}{12}(g_{ij} H_{kp} + g_{jp} H_{ki} + g_{pi} H_{kj})\\
&\qquad + \tfrac{1}{24}(g_{ij} I_{kp} + g_{jp} I_{ki} + g_{pi} I_{kj});
\end{aligned}
$$

where here,

$$I_{ij} = \tfrac{1}{2}\tilde{V}^q T(X_i X_j - X_j X_i) V_q = 2(H_{ij} - {}^{*+}H_{ij});$$

The concomitants in both instances are thus explicitly determined, and one is now in a position to remove the restriction, applied earlier in the analysis, of considering only Lorentz transformations $L(+,+)$: If U is the basic spin matrix corresponding to a transformation (i) $L(+,-)$, (ii) $L(-,+)$, (iii) $L(-,-)$, then

(i)	$U^{-1} = T\tilde{U}T,$	$\tilde{U}TU = -T,$	$U\phi = \phi U;$
(ii)	$U^{-1} = -T\tilde{U}T,$	$\tilde{U}TU = T,$	$U\phi = -\phi U;$
(iii)	$U^{-1} = T\tilde{U}T,$	$\tilde{U}TU = -T,$	$U\phi = -\phi U;$

(cf. (2.2)). In the light of these relations, exactly which of the concomitants are of the pseudo-type corresponding to a particular Lorentz transformation is easily determined, hence extending the analysis to the full Lorentz group. Thus, corresponding to a transformation (i) $L(+,-)$, (ii) $L(-,+)$, (iii) $L(-,-)$,

(i) all of the concomitants are of the pseudo-type;
(ii) ^+F, ^+G_i, $^+H_{ij}$, $^+J_{ij}$, $^+K_{ijk}$, $^+M_{ijk}$ are of the pseudo-type;
(iii) F, G_i, H_{ij}, J_{ij}, K_{ijk}, M_{ijk}, N_{ijkp}, R_{ijkp} are of the pseudo-type.

For example, under a transformation $L(-,-)$, H_{ij} say not only transforms like a tensor of type $[1^2]$, it also undergoes a change in sign.

The concomitants of higher-order spinors can be analysed in an exactly similar manner.

3. *Acknowledgements*

The author wishes to thank Professor D. E. Littlewood for his most helpful advice and guidance during the course of this work. Thanks are also due to the Science Research Council for their award of a Research Studentship during part of the course of this work.

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